

# Gravitational Waves from an Axi-symmetric Source in the Nonsymmetric Gravitational Theory

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## Abstract

We examine gravitational waves in an isolated axi-symmetric reflexion symmetric NGT system. The structure of the vacuum field equations is analyzed and the exact solutions for the field variables in the metric tensor are found in the form of expansions in powers of a radial coordinate. We find that in the NGT axially symmetric case the mass of the system remains constant only if the system is static (as it necessarily is in the case of *spherical* symmetry). If the system radiates, then the mass decreases monotonically and the energy flux associated with waves is positive.

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# 1 Introduction

The present work examines gravitational radiation in the Nonsymmetric Gravitational Theory (NGT) (for a recent detailed review see [1]). We probe the NGT asymptotic behaviour in the wave zone using an *exact* solution.

In General Relativity (GR) gravitational radiation from bounded sources has been studied not only through the linearized theory but also with the use of exact solutions. The latter was done for the general case of a bounded source in asymptotically flat spacetime [2]. It was found that confining the arguments to the axially symmetric case did not cause any essential loss of generality. Since even the relevant GR calculations are very tedious and the level of technical difficulty in the case of NGT increases considerably, we limit ourselves to the axi-symmetric case. The GR gravitational waves from isolated axially symmetric reflexion-symmetric systems were studied in detail in [3]. Since our treatment of the axi-symmetric NGT case is rather parallel, familiarity with this analysis is strongly recommended.

Since the NGT was introduced [4] there have been few analytic solutions of the field equations published. The exact solutions known to date include the spherically symmetric vacuum case [5], the spherically symmetric interior case [6, 7] and Bianchi type I cosmological solutions with and without matter [8, 9].

This, at least in part, follows from the fact that deriving NGT field equations relevant for particular cases of interest is not as technically simple as may be suggested by its superficial similarity to the corresponding GR situations. Firstly, since the underlying geometry is non-Riemannian, neither the fundamental metric tensor  $g_{\mu\nu}$  nor the affine connection is symmetric. This does not constitute a serious problem for the choice of the form of  $g_{\mu\nu}$ , since we can always assume that its nonsymmetric part takes on the isometries of the symmetric part, which in turn has a well defined GR limit. On the other hand, calculating the connection coefficients proves to be a tedious and time consuming exercise, independently of the method chosen. Secondly, the resultant formulae for the nonsymmetric connection coefficients are extremely complicated for all but the simplest forms of the metric, thus becoming unwieldy to use in the derivation of — still more complicated — field equations.

The NGT quantities presented in this paper were derived with the use of symbolic algebraic computation procedures. To this end we have used the symbolic computation system *Maple*.

In Section 2 we briefly summarize the necessary fundamentals of NGT. Section 3 deals with a coordinate system and generalization of the GR metric to the NGT case. Then in Section 4 we expand the metric in negative powers of a suitably chosen radial coordinate and analyze the field equations. The closing section contains conclusions.

Due to their unwieldiness, we retain most of the formulae in the appendices and present the actual calculations only if their tediousness is not forbidding. However, all the results and intermediate steps of the calculations can be made available to the interested reader. In the case of extremely unwieldy quantities the most advisable form of doing that would be the transfer of computer files in the internal *Maple* format.

Throughout this paper we use units in which  $G = c = 1$ .

## 2 NGT Vacuum Field Equations

The NGT Lagrangian without sources takes the form:

$$\mathcal{L} = \sqrt{-g} g^{\mu\nu} R_{\mu\nu}(W), \quad (1)$$

with  $g$  the determinant of  $g_{\mu\nu}$ . The NGT Ricci tensor is defined as:

$$R_{\mu\nu}(W) = W_{\mu\nu,\beta}^{\beta} - \frac{1}{2}(W_{\mu\beta,\nu}^{\beta} + W_{\nu\beta,\mu}^{\beta}) - W_{\alpha\nu}^{\beta} W_{\mu\beta}^{\alpha} + W_{\alpha\beta}^{\beta} W_{\mu\nu}^{\alpha}, \quad (2)$$

and  $W_{\mu\nu}^{\lambda}$  is an unconstrained nonsymmetric connection :

$$W_{\mu\nu}^{\lambda} = W_{(\mu\nu)}^{\lambda} + W_{[\mu\nu]}^{\lambda}. \quad (3)$$

(Throughout this paper parentheses and square brackets enclosing indices stand for symmetrization and antisymmetrization, respectively.) The contravariant nonsymmetric tensor  $g^{\mu\nu}$  is defined in terms of the equation:

$$g^{\mu\nu} g_{\sigma\nu} = g^{\nu\mu} g_{\nu\sigma} = \delta_{\sigma}^{\mu}. \quad (4)$$

If we define the torsion vector as:

$$W_{\mu} \equiv W_{[\mu\nu]}^{\nu} = \frac{1}{2} (W_{\mu\nu}^{\nu} - W_{\nu\mu}^{\nu}), \quad (5)$$

then the connection:

$$\Gamma_{\mu\nu}^{\lambda} = W_{\mu\nu}^{\lambda} + \frac{2}{3} \delta_{\mu}^{\lambda} W_{\nu} \quad (6)$$

is torsion free:

$$\Gamma_\mu \equiv \Gamma_{[\mu\alpha]}^\alpha = 0. \quad (7)$$

Defining now:

$$R_{\mu\nu}(\Gamma) = \Gamma_{\mu\nu,\beta}^\beta - \frac{1}{2}(\Gamma_{(\mu\beta),\nu}^\beta + \Gamma_{(\nu\beta),\mu}^\beta) - \Gamma_{\alpha\nu}^\beta \Gamma_{\mu\beta}^\alpha + \Gamma_{(\alpha\beta)}^\beta \Gamma_{\mu\nu}^\alpha, \quad (8)$$

we can write:

$$R_{\mu\nu}(W) = R_{\mu\nu}(\Gamma) + \frac{2}{3}W_{[\mu,\nu]}, \quad (9)$$

where  $W_{[\mu,\nu]} = \frac{1}{2}(W_{\mu,\nu} - W_{\nu,\mu})$ . Finally, the NGT vacuum field equations can be expressed as:

$$g_{\mu\nu,\sigma} - g_{\rho\nu}\Gamma_{\mu\sigma}^\rho - g_{\mu\rho}\Gamma_{\sigma\nu}^\rho = 0, \quad (10a)$$

$$(\sqrt{-g}g^{[\mu\nu]})_{,\nu} = 0, \quad (10b)$$

$$R_{\mu\nu}(\Gamma) = \frac{2}{3}W_{[\mu,\nu]}. \quad (10c)$$

For the purpose of the analysis of Section 4, it is convenient to decompose  $R_{\mu\nu}$  into standard symmetric and antisymmetric parts:  $R_{(\mu\nu)}$ ,  $R_{[\mu\nu]}$ , and then rewrite the field equation (10c) in the following form:

$$R_{(\mu\nu)}(\Gamma) = 0, \quad (11a)$$

$$R_{[\mu\nu,\rho]}(\Gamma) = 0, \quad (11b)$$

where we used equations (6), (7) and the notation:

$$R_{[\mu\nu,\rho]} = R_{[\mu\nu],\rho} + R_{[\nu\rho],\mu} + R_{[\rho\mu],\nu}. \quad (12)$$

### 3 The Metric

Similarly to GR, the simplest NGT field due to a bounded source would be spherically symmetric. However, the NGT equivalent of Birkhoff's theorem (see e.g. [1]) shows that a spherically symmetric gravitational field in an empty space must be static. Hence, no gravitational radiation escapes into empty space from a pulsating spherically symmetric source.

Following [3] we consider the next simplest case: the field which was initially static and spherically symmetric and eventually becomes such, but undergoes an intermediate non-spherical wave emitting period. Also, spacetime is assumed to be axially symmetric and reflexion-symmetric at all times. Even now, due to the complexity of the field equations, we are forced to use the method

of expansion to examine the problem. This approach, namely expanding in negative powers of a radial coordinate, was also used in the GR analysis [3] and seems to naturally suit a wave problem.

Due to the physical picture sketched above and to the fact that we are interested in the asymptotic behaviour of the field at spatial infinity (in an arbitrary direction from our isolated source) polar coordinates  $x^0 = u, \mathbf{x} = (r, \theta, \phi)$  are the natural choice. The “retarded time”  $u$  has the property that the hypersurfaces  $u = \text{const.}$  are light-like. Detailed discussion of the coordinate systems permissible for investigation of outgoing gravitational waves from isolated systems can be found in [2, 3].

The covariant GR metric tensor corresponding to the situation described above:

$${}^{GR}g_{\mu\nu} = \begin{pmatrix} Vr^{-1}e^{2\beta} - U^2r^2e^{2\gamma} & e^{2\beta} & Ur^2e^{2\gamma} & 0 \\ e^{2\beta} & 0 & 0 & 0 \\ Ur^2e^{2\gamma} & 0 & -r^2e^{2\gamma} & 0 \\ 0 & 0 & 0 & -r^2e^{-2\gamma}\sin^2\theta \end{pmatrix}, \quad (13)$$

with  $U, V, \beta, \gamma$  being functions of  $u, r$  and  $\theta$  was first given in [10].

For any metric in polar coordinates form conditions must be imposed in the neighbourhood of the polar axis,  $\sin\theta = 0$ , to ensure regularity. In the case under consideration we have that, as  $\sin\theta \rightarrow 0$ ,

$$V, \beta, U/\sin\theta, \gamma/\sin^2\theta \quad (14)$$

each is a function of  $\cos\theta$  regular at  $\cos\theta = \pm 1$ .

The NGT generalization of the metric tensor (13) is:

$$g_{\mu\nu} = \begin{pmatrix} Vr^{-1}e^{2\beta} - U^2r^2e^{2\gamma} & e^{2\beta} + \omega & Ur^2e^{2\gamma} + \lambda & 0 \\ e^{2\beta} - \omega & 0 & 0 & 0 \\ Ur^2e^{2\gamma} - \lambda & 0 & -r^2e^{2\gamma} & 0 \\ 0 & 0 & 0 & -r^2e^{-2\gamma}\sin^2\theta \end{pmatrix}, \quad (15)$$

where  $\omega$  and  $\lambda$  are functions of  $u, r$  and  $\theta$ , and the field equation (10b) imposes the following conditions:

$$\mu = 0 \quad : \quad 2\omega^3 - e^{4\beta} [2(1 - 2r\beta_{,r})\omega + r\omega_{,r}] = 0, \quad (16a)$$

$$\mu = 1 \quad : \quad e^{2\beta} [(\cot\theta - 2\gamma_{,\theta})\lambda + \lambda_{,\theta}]$$

$$\begin{aligned}
& +r^2 e^{2\gamma} [(2\beta_{,u} + 2U\beta_{,\theta} - U_{,\theta} - U \cot \theta)\omega - \omega_{,u} - \omega_{,\theta}] \\
& +e^{2\beta} [2(\gamma_{,\theta} - \beta_{,\theta})\omega^2\lambda - \omega^2(\lambda_{,\theta} + \lambda \cot \theta) + \omega\omega_{,\theta}\lambda] \\
& +r^2 e^{-2(\beta-\gamma)} (U_{,\theta} + U \cot \theta)\omega^3 = 0,
\end{aligned} \tag{16b}$$

$$\begin{aligned}
\mu = 2 \quad : \quad & e^{2\beta}(\lambda_{,r} - 2\gamma_{,r}\lambda) - r e^{2\gamma} [(2U - 2rU\beta_{,r} + rU_{,r})\omega + rU\omega_{,r}] \\
& +e^{-2\beta} [2(\gamma_{,r} - \beta_{,r})\omega^2\lambda - \omega^2\lambda_{,r} + \omega\omega_{,r}] \\
& +r e^{-2(\beta-\gamma)} (U + rU_{,r})\omega^3 = 0.
\end{aligned} \tag{16c}$$

We first solve the equation (16a) for  $\omega$ , then substitute the solution into (16c) and solve it for  $\lambda$ .

The solutions are:

$$\omega = e^{2\beta} (1 + C_\omega r^4)^{-1/2}, \tag{17a}$$

$$\lambda = e^{2\gamma} (U + C_\lambda) r^2 (1 + C_\omega r^4)^{-1/2}, \tag{17b}$$

where  $C_\omega(u, \theta)$  and  $C_\lambda(u, \theta)$  are functions of integration satisfying the condition following from the equation (16b):

$$C_{\omega,u} - C_\lambda C_{\omega,\theta} + 2C_\omega (C_{\lambda,\theta} + C_\lambda \cot \theta) = 0. \tag{18}$$

It may be noted here that the functions  $\omega$  and  $\lambda$  must both be present in the metric (15) to ensure its regularity. The case  $\omega \neq 0, \lambda = 0$  leads to a singular  $g^{\mu\nu}$ , while the case  $\omega = 0, \lambda \neq 0$  leads to a solution for  $\lambda$  singular on the polar axis.

Since we consider an asymptotically flat spacetime, it follows from (17b) that to satisfy the boundary conditions we must have  $C_\lambda = 0$ . This in turn, due to the condition (18), restricts  $C_\omega$  to be a constant. To maintain a notation consistent with other NGT results, we write this constant as:

$$C_\omega = l^{-4}.$$

The skew components of the metric are now:

$$\omega = \frac{e^{2\beta} l^2}{(l^4 + r^4)^{1/2}}, \tag{19a}$$

$$\lambda = \frac{e^{2\gamma} U r^2 l^2}{(l^4 + r^4)^{1/2}}. \tag{19b}$$

Finally, the NGT covariant metric tensor we work with takes the form:

$$g_{\mu\nu} = \begin{pmatrix} Vr^{-1}e^{2\beta} - U^2r^2e^{2\gamma} & e^{2\beta}\left(1 + \frac{l^2}{(l^4+r^4)^{1/2}}\right) & e^{2\gamma}Ur^2\left(1 + \frac{l^2}{(l^4+r^4)^{1/2}}\right) & 0 \\ e^{2\beta}\left(1 - \frac{l^2}{(l^4+r^4)^{1/2}}\right) & 0 & 0 & 0 \\ e^{2\gamma}Ur^2\left(1 - \frac{l^2}{(l^4+r^4)^{1/2}}\right) & 0 & -r^2e^{2\gamma} & 0 \\ 0 & 0 & 0 & -r^2e^{-2\gamma}\sin^2\theta \end{pmatrix}. \quad (20)$$

The contravariant metric tensor obtained through (4) is given by:

$$g^{\mu\nu} = \begin{pmatrix} 0 & e^{-2\beta}\left(1 + \frac{l^2}{(l^4+r^4)^{1/2}}\right)^{-1} & 0 & 0 \\ e^{-2\beta}\left(1 - \frac{l^2}{(l^4+r^4)^{1/2}}\right)^{-1} & -Vr^{-1}e^{-2\beta}\frac{l^4+r^4}{r^4} + U^2e^{-4\beta}e^{2\gamma}\frac{l^4}{r^2} & e^{-2\beta}U & 0 \\ 0 & e^{-2\beta}U & -r^{-2}e^{-2\gamma} & 0 \\ 0 & 0 & 0 & -r^{-2}e^{2\gamma}\sin^{-2}\theta \end{pmatrix}. \quad (21)$$

## 4 The Field Equations

Affine connection components for the metric (20) are obtained by solving the system of 64 equations (10a). The list of non-zero components is given in Appendix A. The Ricci tensor is then calculated with the use of (8) and the standard decomposition of its non-zero components into symmetric and antisymmetric parts is performed. The symmetric components  $R_{(\mu\nu)}$  relevant to our calculation can be found in Appendix B. For the sake of convenience in comparing our results to the GR case [3], we have split each  $R_{(\mu\nu)}$  into the GR part  ${}^{GR}R_{\mu\nu}$  and the NGT contributions.

The only non-zero antisymmetric Ricci tensor components for the case at hand are  $R_{[01]}, R_{[02]}, R_{[12]}$ . However, the antisymmetric field equation (11b):

$$R_{[01,2]} = R_{[01],2} + R_{[12],0} + R_{[20],1} = 0, \quad (22)$$

as will be demonstrated shortly, must be trivially satisfied as a consequence of the NGT Bianchi identities. For this reason we do not present these skew components (except for their expanded forms) in this paper.

The seven non-zero symmetric components of the Ricci tensor belong to three categories. The symmetric field equations (11a) for four of them:

$$R_{11} = R_{(12)} = R_{22} = R_{33} = 0, \quad (23)$$

will be called —following the terminology of [3]— “the main equations”. The other two non-trivial symmetric equations:

$$R_{00} = R_{(02)} = 0, \quad (24)$$

we will call —again following [3]— “the supplementary conditions”. The remaining symmetric field equation

$$R_{(01)} = 0$$

is, similarly to the equation (22), identically satisfied due to the Bianchi identities.

The generalized Bianchi identities of NGT can be written [11] (also [12]) in the form:

$$\sqrt{-g}g^{\mu\alpha} (R_{\mu+\alpha-;\rho} - R_{\mu+\rho+;\alpha} - R_{\rho-\alpha-;\mu}) = 0, \quad (25)$$

where we used Einstein’s notation:

$$A^{\beta+};_{\alpha} = A^{\beta}_{,\alpha} + A^{\epsilon}\Gamma_{\epsilon\alpha}^{\beta}, \quad (26a)$$

$$A^{\beta-};_{\alpha} = A^{\beta}_{,\alpha} + A^{\epsilon}\Gamma_{\alpha\epsilon}^{\beta}. \quad (26b)$$

We rewrite (25) in the form:

$$\sqrt{-g}g^{\mu\alpha} (R_{\mu\alpha,\rho} - R_{\mu\rho,\alpha} - R_{\rho\alpha,\mu} + 2R_{(\rho\lambda)}\Gamma_{\mu\alpha}^{\lambda}) = 0. \quad (27)$$

If we now recall the form of the metric (21) and suppose that the four main equations (23) are satisfied, then the Bianchi identities reduce to:

$$\begin{aligned} \rho = 0 : \quad & g^{\mu\alpha}\Gamma_{\mu\alpha}^2 R_{(02)} - g^{(01)} R_{00,1} - g^{11} R_{(01),1} - g^{22} R_{(02),2} \\ & + g^{[12]} (R_{(01),2} + R_{(02),1}) + g^{(12)} (R_{[01],2} + R_{[12],0} + R_{[20],1}) = 0, \end{aligned} \quad (28a)$$

$$\rho = 1 : \quad g^{\mu\alpha}\Gamma_{\mu\alpha}^0 R_{(01)} = 0, \quad (28b)$$

$$\begin{aligned} \rho = 2 : \quad & g^{\mu\alpha}\Gamma_{\mu\alpha}^0 R_{(02)} + g^{[01]} (R_{(01),2} + R_{(02),1}) \\ & + g^{(01)} (R_{[01],2} + R_{[12],0} + R_{[20],1}) = 0. \end{aligned} \quad (28c)$$

From (28b), we see that  $R_{(01)}$  vanishes as a consequence of the main equations. Also, if supplementary conditions are satisfied, then the only non-trivial antisymmetric field equation (22) must hold.



We now rewrite the main equations in the form:

$$0 = R_{11} = \frac{4r^4}{l^4 + r^4} \left[ \left( \frac{\gamma_{,r}^2}{2} - \frac{\beta_{,r}}{r} \right) - \frac{l^4}{r^2(l^4 + r^4)} \right], \quad (29a)$$

$$\begin{aligned} 0 = -2r^2 R_{(12)} &= \left( r^4 e^{2(\gamma-\beta)} U_{,r} \right)_{,r} \\ &\quad - 2r^2 \left[ 2\gamma_{,r}(\gamma_{,\theta} - \cot \theta) - \frac{2\beta_{,\theta}}{r} + \beta_{,r\theta} - \gamma_{,r\theta} \right] \\ &\quad - \frac{2l^4}{l^4 + r^4} e^{2(\gamma-\beta)} r \left[ 3U(1 + 2r\beta_{,r}) - rU_{,r} - 2r^2 U_{,\gamma,r}(1 + r\gamma_{,r}) \right], \end{aligned} \quad (29b)$$

$$\begin{aligned} 0 = R_{22} e^{2(\beta-\gamma)} - r^2 R_3^3 e^{2\beta} &= 2V_{,r} + \frac{r^4}{2} e^{-2(\beta-\gamma)} U_{,r}^2 - r^2 (U \cot \theta + U_{,\theta})_{,r} \\ &\quad - 4r (U \cot \theta + U_{,\theta}) - 2e^{2(\beta-\gamma)} [1 + (3\gamma_{,\theta} - \beta_{,\theta}) \cot \theta \\ &\quad + 2\gamma_{,\theta}(\beta_{,\theta} - \gamma_{,\theta}) - \beta_{,\theta\theta} + \gamma_{,\theta\theta} - \beta_{,\theta}^2] \\ &\quad - \frac{2l^4}{l^4 + r^4} e^{2(\gamma-\beta)} r^2 [U^2(1 + 2r\beta_{,r} - r^2 \gamma_{,r}^2) + rUU_{,r}] \\ &\quad + \frac{2l^4}{l^4 + r^4} \left[ r(U \cot \theta + U_{,\theta}) - \frac{2V}{r} \right], \end{aligned} \quad (29c)$$

$$\begin{aligned} 0 = -r^2 R_3^3 e^{2\beta} &= 2r(r\gamma)_{,ur} + (1 - r\gamma_{,r})(V_{,r} - rU_{,\theta}) - (r\gamma_{,rr} + \gamma_{,r})V \\ &\quad - 2e^{2(\beta-\gamma)} [1 + (3\gamma_{,\theta} - 2\beta_{,\theta}) \cot \theta + 2\gamma_{,\theta}(\beta_{,\theta} - \gamma_{,\theta}) + \gamma_{,\theta\theta}] \\ &\quad + r [2(r\gamma_{,r} + \gamma)_{,\theta} + (r\gamma_{,r} - 3) \cot \theta] U \\ &\quad + r^2 (\gamma_{,\theta} - \cot \theta) U_{,r} \\ &\quad - \frac{2l^4}{l^4 + r^4} e^{-2\beta} \frac{1}{r} \left[ \gamma_{,u} + U(\gamma_{,r} - \cot \theta) + \frac{V}{r^2} (1 - r\gamma_{,r}) \right]. \end{aligned} \quad (29d)$$

The structure of the above equations is analogous to the GR case. The first three equations include only derivatives on the hypersurface  $u = \text{const}$ . The last equation, however, contains differentiation with respect to  $u$ . Thus, if  $\gamma$  is given for some value of  $u$ , then (29a) determines  $\beta$ . In turn (29b) gives  $U$  and then  $V$  follows from (29c). Having all those functions, one can use (29d) to find  $\gamma$  at the later instant of  $u$ . One can then repeat the whole cycle. Thus, if  $\gamma$  is given for some value of  $u$ , the equations (29) determine the temporal evolution of our system except for functions of integration. The latter are arbitrary functions of  $u$  and  $\theta$ , but are independent of  $r$ .

The equations (29) are independent. This is to be expected, since the number of independent NGT field equations must correspond to the number of independent field variables, when the NGT Bianchi identities are taken into account (see e.g. [13]). It may also be noted that the system (29) reduces to its GR counterpart of ref. [3] on substitution  $l^2 = 0$ .

The analysis leading to determining the form of the functions  $U, V, \beta, \gamma$  in our case duplicates

the one given in detail in [3]. The reason is straightforward. Upon inspection of the equations, we find that if the solutions  $U, V, \beta, \gamma$  have forms of an expansion in powers of  $r$ , then the NGT terms of the equations do not contribute to the leading terms. Further, the functions of integration in (29) must remain unchanged, if we are to recover the GR limit upon setting  $l^2 = 0$ . This can be easiest seen in the case of the equation (29a). If  $\gamma$  were to be the same in GR and NGT cases, then the NGT solution for  $\beta$  would have the form:

$$\beta_{NGT} = \frac{1}{4} \ln \left( C_\beta(u, \theta) + \frac{l^4}{r^4} \right) + \beta_{GR} = \frac{1}{4} \ln \left( 1 + \frac{l^4}{r^4} \right) + \beta_{GR},$$

where the second equation takes into account the boundary condition. The leading term in the expansion of  $\beta_{GR}$  is of order  $r^{-2}$  and the highest order NGT contribution goes like  $r^{-4}$ . Therefore, any NGT effect must fall off faster than the GR effects in the wave zone.

The reader interested in the details of determining the functions  $U, V, \beta, \gamma$  is referred to [3]. Here we only summarize the results.

The requirement that the field contain only outgoing radiation at large distances from the source gives the form of  $\gamma$ :

$$\gamma = \frac{f(t-r)}{r} + \frac{g(t-r)}{r^2} + \dots$$

Substituting this into (29), isolating the functions of integration and applying suitable coordinate transformations, gives the following:

$$\begin{aligned} \beta = & - \frac{1}{4} c^2 r^{-2} + \dots \\ & + \frac{1}{4} l^4 r^{-4} - \frac{1}{8} l^8 r^{-8} + \dots, \end{aligned} \quad (30a)$$

$$\gamma = c r^{-1} + \left( C - \frac{1}{6} c^3 \right) r^{-3} + \dots, \quad (30b)$$

$$\begin{aligned} U = & - (c_{,\theta} + 2c \cot \theta) r^{-2} + (2N + 3cc_{,\theta} + 4c^2 \cot \theta) r^{-3} \\ & + \left( \frac{3}{2} C_{,\theta} + 3C \cot \theta - 3cN - 4c^2 c_{,\theta} - 4c^3 \cot \theta \right) r^{-4} + \dots \\ & - \frac{2}{3} (c_{,\theta} + 2c \cot \theta) l^4 r^{-6} + \dots, \end{aligned} \quad (30c)$$

$$\begin{aligned} V = & r \left[ -2M - \left[ N_{,\theta} + N \cot \theta - c_{,\theta}^2 - 4cc_{,\theta} \cot \theta - \frac{1}{2} c^2 (1 + 8 \cot^2 \theta) \right] r^{-1} \right. \\ & - \frac{1}{2} [C_{,\theta\theta} + 3C_{,\theta} \cot \theta - 2C + 6N(c_{,\theta} + 2c \cot \theta) \\ & \quad \left. + 8c(c_{,\theta}^2 + 3cc_{,\theta} + 2c^2 \cot^2 \theta)] r^{-2} + \dots \right. \\ & \left. + \frac{1}{2} l^4 r^{-3} - \frac{1}{6} \left( 6M + \frac{1}{4} c_{,\theta\theta} + \frac{3}{4} c_{,\theta} \cot \theta + c \right) l^4 r^{-4} + \dots, \right. \end{aligned} \quad (30d)$$

where  $c(u, \theta)$ ,  $N(u, \theta)$ ,  $M(u, \theta)$  are the functions of integration and  $C(u, \theta)$  satisfies:

$$4C_{,u} = 2c^2 c_{,u} + 2cM + N \cot \theta - N_{,\theta}.$$

In (30), we isolated the NGT contributions from the rest of the expansion. Clearly, the leading terms are purely GR ones.

One can readily verify the expansions (30) by a direct substitution into (29). The expanded non-zero affine connection components are given in the Appendix C. Again, for the sake of convenience of comparison to the GR results, we split the expansions into the NGT and GR parts. We draw the reader's attention to certain errors occurring in the expanded forms of  $\Gamma_{22}^0$  and  $\Gamma_{12}^1$ , as published in [3].

The expansions (30) considerably simplify the form of the supplementary conditions (24). These extremely complex equations (see Appendix B) reduce now to the leading inverse square-law terms involving only the relations of the functions  $c$ ,  $M$ ,  $N$  and the lower order terms including pure NGT contributions and the coupling of the two. We see that, as we anticipated earlier from the form of the equations (29), the NGT skew contributions decay rapidly with increasing distance from the source and the dominant asymptotic behaviour of the system at spatial infinity is given by  $r^{-2}$  terms. From the latter, we get:

$$M_{,u} = -c_{,u}^2 + \frac{1}{2}(c_{,\theta\theta} + 3c_{,\theta} \cot \theta - 2c)_{,u}, \quad (31a)$$

$$-3N_{,u} = M_{,\theta} + 3cc_{,u\theta} + 4cc_{,u} \cot \theta + c_{,u}c_{,\theta}, \quad (31b)$$

in full agreement with the GR results. We note that the equations (31) are *exact*, if the series expansions are valid.

To complete the discussion of the asymptotic behaviour of the NGT quantities, we expand the only non-zero components of  $R_{[\mu\nu]}$ :

$$R_{[01]} = l^2 \left\{ (4M + 4cc_{,u} + c_{,\theta\theta} + 3c_{,\theta} \cot \theta - 2c) r^{-5} + \dots \right\}, \quad (32a)$$

$$R_{[02]} = l^2 \left\{ 3(c_{,\theta} + 2c \cot \theta)_{,u} r^{-3} + [4c_{,u}(c \cot \theta - c_{,\theta}) + 3c(c_{,\theta} + 2c \cot \theta) + 3c_{,\theta} - M_{,\theta}] r^{-4} + \dots \right\}, \quad (32b)$$

$$R_{[12]} = l^2 \left\{ 3(c_{,\theta} + 2c \cot \theta) r^{-4} - 2[c(3c_{,\theta} + 2c \cot \theta) + 3N] r^{-5} + \dots \right\}. \quad (32c)$$

From (10c), it now follows that the torsion vector  $W_\mu$  has the maximum asymptotic behaviour  $1/r^3$ . This clearly contradicts the result of [14] obtained through a perturbation expansion of the

NGT field equations about a purely Einstein local vacuum background. The reader is referred to [15] for a discussion of the inconsistencies of the latter result.

In order to identify  $M$  and  $N$  one has to turn to the static metrics. We once again refer the reader interested in the details to [3]. After connecting the metric (20) to the static axially symmetric metric in the Weyl form (through a lengthy transformation), one finds that in the static case  $M(u, \theta)$  reduces to the mass  $m$  of the system and is independent of both its arguments. The functions  $N$  and  $C$  are related to the dipole and quadrupole moments, respectively. Also,  $c$  is an arbitrary function of  $\theta$  in the static case. In the time-dependent case the entire information about the temporal evolution of the system is contained in  $c(u, \theta)$ .

Since, in this paper, we are interested only in the physical interpretation of the function  $M$  ( $N$  and  $C$  can be examined with the use of the above mentioned transformation), we limit our analysis here to the following simple exercise. First let us write the non-zero components of the metric tensor (20) using the expansions (30):

$$g_{00} = 1 - \frac{2M}{r} - \frac{1}{r^2}(N_{,\theta} + N \cot \theta) + \dots \\ + \frac{l^4}{r^4} - \frac{l^4}{r^5} \left( 2M + \frac{1}{8}c_{,\theta} \cot \theta + \frac{1}{6}c + \frac{1}{24}c_{,\theta\theta} \right) + \dots, \quad (33a)$$

$$g_{(01)} = 1 - \frac{c^2}{2r^2} + \frac{1}{2r^4} \left( 3Cc - \frac{c^4}{2} \right) + \frac{l^4}{2r^4} + \dots, \quad (33b)$$

$$g_{[01]} = \frac{l^2}{r^2} - \frac{c^2 l^2}{2r^4} + \dots, \quad (33c)$$

$$g_{(02)} = -c_{,\theta} - 2c \cot \theta + \frac{2N + cc_{,\theta}}{r} + \dots, \quad (33d)$$

$$g_{[02]} = -\frac{l^2}{r^2} (c_{,\theta} + 2c \cot \theta) + \frac{l^2}{r^3} (2N + cc_{,\theta}) + \dots, \quad (33e)$$

$$g_{22} = -r^2 - c^2 - 2cr - \frac{1}{r} \left( 2C - \frac{1}{3}c^3 \right) + \dots, \quad (33f)$$

$$g_{33} = \sin^2 \theta \left[ -r^2 - c^2 + 2cr + \frac{1}{r} \left( 2C - \frac{1}{3}c^3 \right) + \dots \right]. \quad (33g)$$

We confine ourselves to initial and final static systems. Suppose that the system is in a dynamic period, which can be physically interpreted only as the emission of waves, between the moments  $u_i$  and  $u_f$ . Thus, for  $u \leq u_i$  and  $u \geq u_f$  the function  $c$  does not vary. The static limit  $c(u, \theta) \rightarrow c_s(\theta)$  of (33) gives the equivalent of the static Weyl metric in our case. However, to demonstrate the physical interpretation of  $M$ , we simplify the situation further. We can scale the function  $c$  for either one of the static periods to be  $c = 0$  (forsaking the  $\theta$  dependence of  $c$  limits us here to a static spherically symmetric system). We now remove the terms containing the functions  $N$  and  $C$

from (33). We have a *post factum* justification for doing that from the interpretation of  $N$  and  $C$  as multipole moments. Since there is no radiation during the static period, then  $N = C = 0$ . The metric (33) tends now to its static spherically symmetric limit:

$$g_{00} = 1 - \frac{2M_s}{r} + \frac{l^4}{r^4} - \frac{2M_sl^4}{r^5}, \quad (34a)$$

$$g_{(01)} = 1 + \frac{l^4}{2r^4}, \quad (34b)$$

$$g_{[01]} = \frac{l^2}{r^2}, \quad (34c)$$

$$g_{(02)} = g_{[02]} = 0, \quad (34d)$$

$$g_{22} = -r^2, \quad (34e)$$

$$g_{33} = -r^2 \sin^2 \theta, \quad (34f)$$

where by  $M_s$  we denote the static limit of  $M$ .

Now a coordinate transformation from our retarded time  $u$  to the usual time coordinate  $t = u + r$  converts (34) into the NGT static spherically symmetric metric [1, 5]:

$$ds^2 = \left(1 + \frac{l^4}{r^4}\right) \left(1 - \frac{2M_s}{r}\right) dt^2 - \left(1 - \frac{2M_s}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (35)$$

with

$$g_{[01]} = \frac{l^2}{r^2}.$$

Thus, the static spherically symmetric limit  $M_s$  of the “mass aspect”  $M(u, \theta)$  can only be interpreted as the mass of the system.

If we define the mass  $m(u)$  of the system as the mean value of  $M(u, \theta)$  over the sphere:

$$m(u) = \frac{1}{2} \int_0^\pi M(u, \theta) \sin \theta d\theta, \quad (36)$$

then  $c(u, \theta)$  completely determines the time evolution of the mass  $m(u)$ . Integrating (31a) and noticing that the second term in it does not contribute to the integral due to the conditions (14), we get:

$$m_{,u} = \frac{dm}{du} = -\frac{1}{2} \int_0^\pi c_{,u}^2 \sin \theta d\theta. \quad (37)$$

Since we discussed here systems whose initial and final states are static, the physical interpretation of  $m(u)$  as the mass of the system is unambiguous. Analogously to the GR case the main result is as follows:

*The mass of an axially symmetric NGT system is constant only if the system remains static. If the system evolves in time (emits waves), the mass decreases monotonically.*

Since radiation is the only energy loss mechanism available to the system, the above proves that gravitational waves emitted by an axi-symmetric reflexion symmetric NGT source compatible with the metric (20) carry positive energy or, in other words, the flux of gravitational energy in NGT is positive.

## 5 Conclusions

We have proved that an NGT axi-symmetric system emitting gravitational waves has the usual GR-like asymptotic behaviour in the wave zone. The NGT contributions to the physical quantities decay rapidly with the distance from the source and the energy flux at spatial infinity is necessarily positive.

Whether a similar result could be obtained in the general NGT case without any symmetries remains to be verified, although it is expected that a proof of this would be similar to the one in [2].

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## A List of non-zero affine connection components

$$\begin{aligned}\Gamma_{00}^0 &= GR\Gamma_{00}^0 + \frac{4l^4}{r^2(l^4 + r^4)} \left( U^2 r^3 e^{-2(\beta-\gamma)} - V \right), \\ GR\Gamma_{00}^0 &= 2\beta_{,u} + r^2 e^{2(\beta-\gamma)} U \left( U_{,r} + \frac{U}{r} + \gamma_{,r} U \right) - \frac{\beta_{,r} V}{r} - \frac{V_{,r}}{2r} + \frac{V}{2r^3},\end{aligned}\tag{38}$$

$$\Gamma_{01}^0 = -\frac{2l^2}{r(l^4 + r^4)} \left( l^2 - \sqrt{l^4 + r^4} \right),\tag{39}$$

$$\Gamma_{02}^0 = GR\Gamma_{02}^0 - \frac{l^2 r}{l^4 + r^4} \left( 3l^2 + \sqrt{l^4 + r^4} \right) U e^{-2(\beta-\gamma)},$$

$$GR\Gamma_{02}^0 = \beta_{,\theta} - r^2 e^{-2(\beta-\gamma)} \left( \frac{U_{,r}}{2} + \frac{U}{r} + \gamma_{,r} U \right),\tag{40}$$

$$\Gamma_{10}^0 = -\frac{2l^2}{r(l^4 + r^4)} \left( l^2 + \sqrt{l^4 + r^4} \right),\tag{41}$$

$$\Gamma_{20}^0 = GR\Gamma_{02}^0 - \frac{l^2 r}{l^4 + r^4} \left( 3l^2 - \sqrt{l^4 + r^4} \right) U e^{-2(\beta-\gamma)},\tag{42}$$

$$\Gamma_{22}^0 = GR\Gamma_{22}^0 = r e^{-2(\beta-\gamma)} (1 + r\gamma_{,r}),\tag{43}$$

$$\Gamma_{33}^0 = GR\Gamma_{33}^0 = r e^{-2(\beta+\gamma)} \sin^2 \theta (1 - r\gamma_{,r}),\tag{44}$$

$$\begin{aligned}\Gamma_{00}^1 &= GR\Gamma_{00}^1 + \frac{4l^4}{r^3(l^4 + r^4)} V^2 + \frac{l^4}{r(2l^4 + r^4)} UV_{,\theta} + \frac{2l^4 r^3}{l^4 + r^4} U^4 e^{-4(\beta-\gamma)} \\ &\quad + \frac{l^4 r^4}{2l^4 + r^4} U^3 U_{,r} e^{-4(\beta-\gamma)} - \frac{3l^4(3l^4 + r^4)}{(l^4 + r^4)(2l^4 + r^4)} V U^2 e^{-2(\beta-\gamma)} \\ &\quad + \frac{r l^4}{2l^4 + r^4} e^{-2(\beta-\gamma)} \left[ 2V U^2 (\gamma - \beta)_{,r} + U(U_{,r} V - U V_{,r}) \right. \\ &\quad \left. + 2r U^3 (\beta - \gamma)_{,\theta} + 4r U^2 (\beta - \gamma)_{,u} + 2r U(U_{,u} + U U_{,\theta}) \right],\end{aligned}$$

$$\begin{aligned}GR\Gamma_{00}^1 &= \frac{V_{,u}}{2r} - \frac{\beta_{,u} V}{r} - \frac{U V_{,\theta}}{2r} - \frac{\beta_{,\theta} U V}{r} + \frac{V V_{,r}}{2r^2} + \frac{\beta_{,r} V^2}{r^2} - \frac{V^2}{2r^3} \\ &\quad + r^2 e^{-2(\beta-\gamma)} \left[ U^2 \left( U_{,\theta} + \gamma_{,\theta} U - \frac{V}{r^2} - \frac{\gamma_{,r} V}{r} + \gamma_{,u} \right) - \frac{U U_{,r} V}{r} \right],\end{aligned}\tag{45}$$

$$\begin{aligned}\Gamma_{01}^1 &= GR\Gamma_{01}^1 + \frac{2l^4}{l^4 + r^4} \left[ \frac{2V}{r^2} - r U^2 e^{-2(\beta-\gamma)} \right] + \frac{l^2}{\sqrt{l^4 + r^4}} \left( U \beta_{,\theta} - \frac{2V}{r^2} \right) \\ &\quad - \frac{l^2}{\sqrt{l^4 + r^4}} r e^{-2(\beta-\gamma)} \left[ \frac{l^4}{l^4 + r^4} U^2 + r U^2 \gamma_{,r} + \frac{1}{2} r U U_{,r} \right],\end{aligned}$$

$$GR\Gamma_{01}^1 = \frac{V_{,r}}{2r} - \frac{V}{2r^2} + \frac{\beta_{,r} V}{r} - \beta_{,\theta} U - \frac{1}{2} r^2 e^{-2(\beta-\gamma)} U U_{,r},\tag{46}$$

$$\begin{aligned}\Gamma_{02}^1 &= GR\Gamma_{02}^1 - \frac{l^4}{2l^4 + r^4} V_{,\theta} - \frac{l^4}{2l^4 + r^4} \left( \frac{2}{l^4 + r^4} r^7 U^3 + \frac{1}{2} U^2 U_{,r} \right) e^{-4(\beta-\gamma)} \\ &\quad + \frac{l^4}{2(2l^4 + r^4)} e^{-2(\beta-\gamma)} \left\{ U V [2r(\beta - \gamma)_{,r} - 3] + r(U V_{,r} - U_{,r} V) \right. \\ &\quad \left. + 2r^2 [U^2 (\gamma - \beta)_{,\theta} + U_{,u} + U U_{,\theta} - 2U(\beta - \gamma)_{,u}] \right\}\end{aligned}$$

$$\begin{aligned}
& + \frac{l^4}{(l^4 + r^4)(2l^4 + r^4)} r^3 UV e^{-2(\beta-\gamma)} - \frac{l^2 \sqrt{l^4 + r^4}}{2l^4 + r^4} V_{,\theta} \\
& + \frac{l^6}{(2l^4 + r^4) \sqrt{l^4 + r^4}} r^4 U^2 (r U_{,r} + 2U) e^{-4(\beta-\gamma)} \\
& + \frac{l^2 \sqrt{l^4 + r^4}}{2l^4 + r^4} e^{-2(\beta-\gamma)} \left\{ 2r^3 [U(U_{,\theta} + U\gamma_{,\theta}) + U(\gamma - 2\beta)_{,u} + U_{,u}] \right. \\
& \quad \left. + r^2 (UV_{,r} - U_{,r}V) + 2r^2 UV\beta_{,r} - rUV \right\} \\
& + \frac{l^6}{(2l^4 + r^4) \sqrt{l^4 + r^4}} e^{-2(\beta-\gamma)} \left[ 2U(V + r^2 V_{,r}) - 2r^3 U(U\beta_{,\theta} - \gamma_{,u}) \right] \\
& + \frac{2l^2}{\sqrt{l^4 + r^4}} r UV e^{-2(\beta-\gamma)} + \frac{4l^2(l^4 + r^4)^{3/2}}{r(2l^4 + r^4)} U^2 \beta_{,\theta} e^{-2(\beta-\gamma)}, \\
GR\Gamma_{02}^1 &= \frac{V_{,\theta}}{2r} + r^2 e^{-2(\beta-\gamma)} \left[ U \left( \frac{V}{r^2} + \frac{\gamma_{,r}V}{r} - \gamma_{,u} - U_{,\theta} - \gamma_{,\theta}U \right) + \frac{U_{,r}V}{2r} \right], \tag{47}
\end{aligned}$$

$$\begin{aligned}
\Gamma_{10}^1 &= GR\Gamma_{01}^1 + \frac{2l^4}{l^4 + r^4} \left[ \frac{2V}{r^2} - rU^2 e^{-2(\beta-\gamma)} \right] - \frac{l^2}{\sqrt{l^4 + r^4}} \left( U\beta_{,\theta} - \frac{2V}{r^2} \right) \\
& + \frac{l^2}{\sqrt{l^4 + r^4}} r e^{-2(\beta-\gamma)} \left[ \frac{l^4}{l^4 + r^4} U^2 + rU^2 \gamma_{,r} + \frac{1}{2} r U U_{,r} \right], \tag{48}
\end{aligned}$$

$$\begin{aligned}
\Gamma_{11}^1 &= GR\Gamma_{11}^1 + \frac{4l^4}{(l^4 + r^4)r}, \\
GR\Gamma_{11}^1 &= 2\beta_{,r}, \tag{49}
\end{aligned}$$

$$\begin{aligned}
\Gamma_{12}^1 &= GR\Gamma_{12}^1 + e^{-2(\beta-\gamma)} U r \left[ \frac{l^2}{\sqrt{l^4 + r^4}} r \gamma_{,r} - \frac{3l^4}{l^4 + r^4} \right], \\
GR\Gamma_{12}^1 &= \beta_{,\theta} + \frac{1}{2} r^2 e^{-2(\beta-\gamma)} U_{,r}, \tag{50}
\end{aligned}$$

$$\begin{aligned}
\Gamma_{20}^1 &= GR\Gamma_{02}^1 - \frac{l^4}{2l^4 + r^4} V_{,\theta} - \frac{l^4}{2l^4 + r^4} \left( \frac{2}{l^4 + r^4} r^7 U^3 + \frac{1}{2} U^2 U_{,r} \right) e^{-4(\beta-\gamma)} \\
& + \frac{l^4}{2(2l^4 + r^4)} e^{-2(\beta-\gamma)} \left\{ UV[2r(\beta - \gamma)_{,r} - 3] + r(UV_{,r} - U_{,r}V) \right. \\
& \quad \left. + 2r^2 [U^2(\gamma - \beta)_{,\theta} + U_{,u} + UU_{,\theta} - 2U(\beta - \gamma)_{,u}] \right\} \\
& + \frac{l^4}{(l^4 + r^4)(2l^4 + r^4)} r^3 UV e^{-2(\beta-\gamma)} + \frac{l^2 \sqrt{l^4 + r^4}}{2l^4 + r^4} V_{,\theta} \\
& - \frac{l^6}{(2l^4 + r^4) \sqrt{l^4 + r^4}} r^4 U^2 (r U_{,r} + 2U) e^{-4(\beta-\gamma)} \\
& - \frac{l^2 \sqrt{l^4 + r^4}}{2l^4 + r^4} e^{-2(\beta-\gamma)} \left\{ 2r^3 [U(U_{,\theta} + U\gamma_{,\theta}) + U(\gamma - 2\beta)_{,u} + U_{,u}] \right. \\
& \quad \left. + r^2 (UV_{,r} - U_{,r}V) + 2r^2 UV\beta_{,r} - rUV \right\} \\
& - \frac{l^6}{(2l^4 + r^4) \sqrt{l^4 + r^4}} e^{-2(\beta-\gamma)} \left[ 2U(V + r^2 V_{,r}) - 2r^3 U(U\beta_{,\theta} - \gamma_{,u}) \right]
\end{aligned}$$



$$-\frac{2l^2}{\sqrt{l^4+r^4}}rUVe^{-2(\beta-\gamma)} - \frac{4l^2(l^4+r^4)^{3/2}}{r(2l^4+r^4)}U^2\beta_{,\theta}e^{-2(\beta-\gamma)}, \quad (51)$$

$$\Gamma_{21}^1 = GR\Gamma_{12}^1 - e^{-2(\beta-\gamma)}Ur \left[ \frac{l^2}{\sqrt{l^4+r^4}}r\gamma_{,r} - \frac{3l^4}{l^4+r^4} \right], \quad (52)$$

$$\begin{aligned} \Gamma_{22}^1 &= GR\Gamma_{22}^1 + \frac{l^4}{l^4+r^4}U^2r^3e^{-4(\beta-\gamma)}, \\ GR\Gamma_{22}^1 &= r^2e^{-2(\beta-\gamma)} \left( \gamma_{,u} + U_{,\theta} + \gamma_{,\theta}U - \frac{V}{r^2} - \frac{\gamma_{,r}V}{r} \right), \end{aligned} \quad (53)$$

$$\Gamma_{33}^1 = GR\Gamma_{33}^1 = r^2 \sin^2 \theta e^{-2(\beta+\gamma)} \left( -\gamma_{,u} + U \cot \theta - \gamma_{,\theta}U - \frac{V}{r^2} + \frac{\gamma_{,r}V}{r} \right), \quad (54)$$

$$\begin{aligned} \Gamma_{00}^2 &= GR\Gamma_{00}^2 - \frac{l^4}{l^4+r^4} \frac{2UV}{r^2} - \frac{l^4}{2l^4+r^4} \frac{V_{,\theta}}{r^3} e^{2(\beta-\gamma)} \\ &\quad + \left[ \frac{l^4}{l^4+r^4} rU^3 - \frac{l^4}{2l^4+r^4} r^2U^2U_{,r} \right] e^{-2(\beta-\gamma)} \\ &\quad - \frac{l^4}{(2l^4+r^4)r^2} \left[ 3UV - r(UV_{,r} - U_{,r}V) - 2rUV(\beta-\gamma)_{,r} \right. \\ &\quad \left. + 4r^2U(\beta-\gamma)_{,u} + 2r^2U^2(\beta-\gamma)_{,\theta} - 2r^2(U_{,u} + UU_{,\theta}) \right], \\ GR\Gamma_{00}^2 &= -U_{,u} + U \left( 2\beta_{,u} - 2\gamma_{,u} - U_{,\theta} - \gamma_{,\theta}U + \frac{V}{2r^2} - \frac{V_{,r}}{2r} - \frac{\beta_{,r}V}{r} \right) \\ &\quad + e^{2(\beta-\gamma)} \frac{V_{,\theta} + 2\beta_{,\theta}V}{2r^3} + r^2e^{-2(\beta-\gamma)}U^2 \left( U_{,r} + \frac{U}{r} + \gamma_{,r}U \right), \end{aligned} \quad (55)$$

$$\begin{aligned} \Gamma_{01}^2 &= GR\Gamma_{01}^2 + \frac{l^2}{\sqrt{l^4+r^4}} \frac{2\beta_{,\theta}}{r^2} e^{2(\beta-\gamma)} - \frac{l^2}{\sqrt{l^4+r^4}} \left[ \frac{2U}{r}(1+r\gamma_{,r}) + U_{,r} \right] \\ &\quad - \frac{l^2}{\sqrt{l^4+r^4}} \left( \frac{l^2}{\sqrt{l^4+r^4}} - 1 \right)^2 \frac{2U}{r}, \\ GR\Gamma_{01}^2 &= -\frac{U_{,r}}{2} - \frac{U}{r} - \gamma_{,r}U + \frac{\beta_{,\theta}}{r^2} e^{2(\beta-\gamma)}, \end{aligned} \quad (56)$$

$$\begin{aligned} \Gamma_{02}^2 &= GR\Gamma_{02}^2 - \frac{l^2}{\sqrt{l^4+r^4}} \left[ \frac{V\gamma_{,r}}{r} - \gamma_{,u} - U\beta_{,\theta} + V \right] \\ &\quad + \frac{l^2}{2l^4+r^4} r^2UU_{,r}e^{-2(\beta-\gamma)} \\ &\quad - \frac{l^2}{\sqrt{l^4+r^4}} \left( \frac{l^2}{\sqrt{l^4+r^4}} + 1 \right)^2 rU^2e^{-2(\beta-\gamma)}, \\ GR\Gamma_{02}^2 &= \gamma_{,u} + \beta_{,\theta}U - r^2e^{-2(\beta-\gamma)}U \left( \frac{U_{,r}}{2} + \frac{U}{r} + \gamma_{,r}U \right), \end{aligned} \quad (57)$$

$$\begin{aligned} \Gamma_{10}^2 &= GR\Gamma_{01}^2 - \frac{l^2}{\sqrt{l^4+r^4}} \frac{2\beta_{,\theta}}{r^2} e^{2(\beta-\gamma)} + \frac{l^2}{\sqrt{l^4+r^4}} \left[ \frac{2U}{r}(1+r\gamma_{,r}) + U_{,r} \right] \\ &\quad + \frac{l^2}{\sqrt{l^4+r^4}} \left( \frac{l^2}{\sqrt{l^4+r^4}} - 1 \right)^2 \frac{2U}{r}, \end{aligned} \quad (58)$$

$$\Gamma_{12}^2 = GR\Gamma_{12}^2 - \frac{l^2}{\sqrt{l^4+r^4}} \left( \frac{1}{r} + \gamma_{,r} \right),$$

$${}^{GR}\Gamma_{12}^2 = \frac{1}{r} + \gamma_{,r}, \quad (59)$$

$$\begin{aligned} \Gamma_{20}^2 &= {}^{GR}\Gamma_{02}^2 + \frac{l^2}{\sqrt{l^4 + r^4}} \left[ \frac{V\gamma_{,r}}{r} - \gamma_{,u} - U\beta_{,\theta} + V \right] \\ &\quad - \frac{l^2}{2l^4 + r^4} r^2 U U_{,r} e^{-2(\beta-\gamma)} \\ &\quad + \frac{l^2}{\sqrt{l^4 + r^4}} \left( \frac{l^2}{\sqrt{l^4 + r^4}} - 1 \right)^2 r U^2 e^{-2(\beta-\gamma)}, \end{aligned} \quad (60)$$

$$\Gamma_{21}^2 = {}^{GR}\Gamma_{12}^2 + \frac{l^2}{\sqrt{l^4 + r^4}} \left( \frac{1}{r} + \gamma_{,r} \right), \quad (61)$$

$$\Gamma_{22}^2 = {}^{GR}\Gamma_{22}^2 = \gamma_{,\theta} + r^2 e^{-2(\beta-\gamma)} U \left( \frac{1}{r} + \gamma_{,r} \right), \quad (62)$$

$$\begin{aligned} \Gamma_{33}^2 &= {}^{GR}\Gamma_{33}^2 = r^2 \sin^2 \theta e^{2(\beta-\gamma)} U \left( \frac{1}{r} - \gamma_{,r} \right) \\ &\quad - e^{-4\gamma} \sin^2 \theta (\cot \theta - \gamma_{,\theta}), \end{aligned} \quad (63)$$

$$\Gamma_{03}^3 = {}^{GR}\Gamma_{03}^3 - \frac{l^2}{\sqrt{l^4 + r^4}} \left[ \gamma_{,u} + \left( \frac{1}{r} - \gamma_{,r} \right) \left( \frac{V}{r} - r^2 U^2 e^{-2(\beta-\gamma)} \right) \right],$$

$${}^{GR}\Gamma_{03}^3 = -\gamma_{,u}, \quad (64)$$

$$\Gamma_{13}^3 = {}^{GR}\Gamma_{13}^3 - \frac{l^2}{\sqrt{l^4 + r^4}} \left( \frac{1}{r} - \gamma_{,r} \right),$$

$${}^{GR}\Gamma_{13}^3 = \frac{1}{r} - \gamma_{,r}, \quad (65)$$

$$\Gamma_{23}^3 = {}^{GR}\Gamma_{23}^3 - \frac{l^2}{\sqrt{l^4 + r^4}} \left( \frac{1}{r} - \gamma_{,r} \right) r^2 U e^{-2(\beta-\gamma)},$$

$${}^{GR}\Gamma_{23}^3 = \cot \theta - \gamma_{,\theta}, \quad (66)$$

$$\Gamma_{30}^3 = {}^{GR}\Gamma_{03}^3 + \frac{l^2}{\sqrt{l^4 + r^4}} \left[ \gamma_{,u} + \left( \frac{1}{r} - \gamma_{,r} \right) \left( \frac{V}{r} - r^2 U^2 e^{-2(\beta-\gamma)} \right) \right], \quad (67)$$

$$\Gamma_{31}^3 = {}^{GR}\Gamma_{13}^3 + \frac{l^2}{\sqrt{l^4 + r^4}} \left( \frac{1}{r} - \gamma_{,r} \right), \quad (68)$$

$$\Gamma_{32}^3 = {}^{GR}\Gamma_{23}^3 + \frac{l^2}{\sqrt{l^4 + r^4}} \left( \frac{1}{r} - \gamma_{,r} \right) r^2 U e^{-2(\beta-\gamma)}. \quad (69)$$

## B List of non-zero symmetric Ricci tensor components

$$\begin{aligned}
R_{00} = {}^{GR} R_{00} & - \frac{l^4}{2l^4 + r^4} e^{2(\beta-\gamma)} \frac{1}{r^3} \left[ 2V_{,\theta}(\beta-\gamma)_{,\theta} + V_{,\theta} \cot \theta + V_{,\theta\theta} \right] \\
& + \frac{l^4}{r(2l^4 + r^4)} \left\{ 2UV[(\beta-\gamma)_{,\theta} + (\beta-\gamma) \cot \theta]_{,r} - 4r(U_{,\theta} + U \cot \theta)(\beta-\gamma)_{,u} \right. \\
& \quad - 2r(2UU_{,\theta} + U^2 \cot \theta)(\beta-\gamma)_{,\theta} + 2r(U_{,u\theta} + U_{,\theta}^2 + UU_{,\theta\theta} \\
& \quad + 2(U_{,\theta}V + 2UV_{,\theta})(\beta-\gamma)_{,r} + (UV_{,r} - U_{,r}V) \cot \theta \\
& \quad + 2r(U_{,u} + UU_{,\theta}) \cot \theta - 4rU(\beta-\gamma)_{,u\theta} \\
& \quad \left. - 2rU^2(\beta-\gamma)_{,\theta\theta} - (U_{,r}V)_{,\theta} + U_{,\theta}V_{,r} + 2UV_{,r\theta} \right\} \\
& - \frac{l^4}{(l^4 + r^4)^2} \frac{l^4 + 9r^4}{r^4} V^2 - \frac{l^4}{(l^4 + r^4)(2l^4 + r^4)} \frac{7l^4 + 5r^4}{r^2} (U \cot \theta + U_{,\theta})V \\
& - \frac{l^4}{(l^4 + r^4)(2l^4 + r^4)^2} \frac{8l^8 + 14l^4r^4 + 7r^8}{r^2} UV_{,\theta} \\
& - \frac{l^4}{r^2(l^4 + r^4)} \left[ V_{,u} - 2r^2\gamma_{,u}^2 + 2V(2r\gamma_{,u}\gamma_{,r} - \beta_{,u} - V\gamma_{,r}^2 + U\beta_{,\theta}) \right. \\
& \quad \left. + \frac{V(2V\beta_{,r} - 3V_{,r})}{r} \right] \\
& + \frac{l^4}{2l^4 + r^4} r^2 U e^{-2(\beta-\gamma)} \left\{ 6UU_{,r}(\beta-\gamma)_{,\theta} + 4[U(\beta-\gamma)_{,u}]_{,r} + 2U^2(\beta-\gamma)_{,r\theta} \right. \\
& \quad \left. - U(3U_{,\theta} + U \cot \theta)_{,r} - 2(U_{,ur} + 2U_{,r}U_{,\theta}) \right\} \\
& - \frac{l^4}{2l^4 + r^4} r U e^{-2(\beta-\gamma)} \left\{ UV_{,rr} - U_{,rr}V + [UV(\beta-\gamma)_{,r}]_{,r} \right\} \\
& + \frac{2l^4}{l^4 + r^4} r U^2 e^{-2(\beta-\gamma)} (2r\gamma_{,u}\gamma_{,r} + U \cot \theta - 2V\gamma_{,r}^2) \\
& + \frac{2l^4}{(l^4 + r^4)^2(2l^4 + r^4)^2} (r^{12} + 10l^{12} + 17l^4r^8 + 29l^8r^4) \frac{V}{r} U^2 e^{-2(\beta-\gamma)} \\
& - \frac{2l^4}{(l^4 + r^4)^2(2l^4 + r^4)^2} (l^{12} - 3r^{12} + l^4r^8 + 8l^8r^4) r\gamma_{,u} U^2 e^{-2(\beta-\gamma)} \\
& + \frac{2l^4}{(l^4 + r^4)(2l^4 + r^4)^2} \left[ 2r^9(UU_{,\theta} + U_{,u} - 2U\beta_{,u}) - r(4l^8 + 4l^4r^4 - r^8)U^2\gamma_{,\theta} \right. \\
& \quad - (4l^8 + 2l^4r^4 - r^8)UV_{,r} + (10l^8 + 11l^4r^4 + 5r^8)UV\beta_{,r} \\
& \quad \left. + l^4(2l^4 + 3r^4)U_{,r}V + (2l^8 + l^4r^4 - 2r^8)UV\gamma_{,r} \right] U e^{-2(\beta-\gamma)} \\
& + \frac{16l^8}{(2l^4 + r^4)^2} r U^3 \beta_{,\theta} e^{-2(\beta-\gamma)} - \frac{2l^4}{l^4 + r^4} r^3 U^4 e^{-4(\beta-\gamma)} (2\beta_{,r} - r\gamma_{,r}^2) \\
& + \left[ \frac{2l^4}{(2l^4 + r^4)^2} (6l^4 + r^4) r^3 U_{,r} - \frac{2l^4(2l^4 + r^4)}{(l^4 + r^4)^2} r U \right] U^3 e^{-4(\beta-\gamma)}
\end{aligned}$$

$$\begin{aligned}
& + \frac{l^4}{2l^4 + r^4} r^4 U^2 e^{-4(\beta-\gamma)} \left[ UU_{,rr} + 2U_{,r}^2 - 2UU_{,r}(\beta - \gamma)_{,r} \right], \\
{}^{GR}R_{00} = & - 2U(\beta_{,u\theta} - \gamma_{,u\theta}) - 2U_{,\theta}(\beta_{,u} - \gamma_{,u}) + U_{,u\theta} + UU_{,\theta\theta} + U_{,\theta}^2 \\
& - 2UU_{,\theta}(\beta_{,\theta} - \gamma_{,\theta}) + U^2(2\beta_{,\theta}^2 - 2\beta_{,\theta}\gamma_{,\theta} + \gamma_{,\theta\theta}) + 2\gamma_{,u}^2 \\
& - \cot \theta \left[ 2U(\beta_{,u} - \gamma_{,u}) - U_{,u} - UU_{,\theta} - U^2\gamma_{,\theta} \right] \\
& + e^{-2(\beta-\gamma)} \left[ 4UU_{,r}V + U^2(V\gamma_{,r} + V_{,r}) \right] - \frac{1}{2r}(U_{,r}V_{,\theta} - U_{,\theta}V_{,r} - UV_{,r}\cot \theta) \\
& + \frac{1}{r} [2V(\beta_{,ur} + U\beta_{,r\theta}) + 2UV_{,\theta}(\beta_{,r} - \gamma_{,r}) + U_{,\theta}V\beta_{,r} \\
& \quad + U(V_{,r}\beta_{,\theta} + V_{,r\theta} + V\beta_{,r}\cot \theta)] - \frac{1}{2r^2}V(U_{,\theta} + V_{,rr} + U\cot \theta) \\
& - \frac{1}{r^2} [2U(V_{,\theta} - V\beta_{,\theta}) + V(V\beta_{,rr} + V_{,r}\beta_{,r} - 2\beta_{,u}) + V_{,u}] \\
& - \frac{1}{2r^3}e^{2(\beta-\gamma)} [V_{,\theta\theta} + 2V\beta_{,\theta\theta} + (2\beta_{,\theta} - 2\gamma_{,\theta} + \cot \theta)(V_{,\theta} + 2V\beta_{,\theta})] \\
& + re^{-2(\beta-\gamma)} [UV(U_{,rr} - 2U_{,r}\beta_{,r} + 2U_{,r}\gamma_{,r}) \\
& \quad + U^2(V\gamma_{,rr} + V_{,r}\gamma_{,r} - 3U_{,\theta} - 2\gamma_{,u}) - U^3(\cot \theta + 2\gamma_{,\theta}) + \frac{U_{,r}^2V}{2}] \\
& + r^2e^{-2(\beta-\gamma)} \left[ -2U^3\gamma_{,r\theta} + U^2(2U_{,r}\beta_{,\theta} - 3U_{,r}\gamma_{,\theta} - U_{,\theta}\gamma_{,r} - 2\gamma_{,ur} - 2U_{,r\theta}) \right. \\
& \quad \left. - U^2(U_{,r} + U\gamma_{,r})\cot \theta - U(U_{,ur} + 2U_{,r}U_{,\theta}) + 2UU_{,r}(\beta_{,u} - \gamma_{,u}) \right] \\
& + \frac{1}{2}r^4e^{-4(\beta-\gamma)}U^2U_{,r}^2,
\end{aligned} \tag{71}$$

$$\begin{aligned}
R_{(01)} = {}^{GR}R_{01} & - \frac{l^4}{l^4 + r^4} \left[ \frac{2U_{,\theta}}{r} + \frac{2U}{r}(\beta_{,\theta} + \cot \theta) - \frac{2V\gamma_{,r}^2}{r} + 2\gamma_{,u}\gamma_{,r} + \frac{2V\beta_{,r} - 3V_{,r}}{r^2} + \frac{V}{r^3} \right] \\
& - \frac{8l^4}{(l^4 + r^4)^2}rV - \frac{2l^8}{(l^4 + r^4)^2}U^2e^{-2(\beta-\gamma)} \\
& - \frac{l^4}{l^4 + r^4}e^{-2(\beta-\gamma)} \left[ 2U^2(r^2\gamma_{,r}^2 - 2r\beta_{,r} - 1) + rUU_{,r} \right],
\end{aligned} \tag{72}$$

$$\begin{aligned}
{}^{GR}R_{01} = & - U(\beta_{,r\theta} - \gamma_{,r\theta}) - \left( \frac{U_{,r}}{2} + U\gamma_{,r} \right) \cot \theta - U_{,r}\beta_{,\theta} - U_{,\theta}\gamma_{,r} - 2\gamma_{,u}\gamma_{,r} \\
& - \frac{U_{,r\theta}}{2} - 2\beta_{,ur} + \frac{V\beta_{,r}}{r^2} + \frac{1}{r^2}e^{2(\beta-\gamma)} (2\beta_{,\theta}^2 - 2\beta_{,\theta}\gamma_{,\theta} + \beta_{,\theta}\cot \theta + \beta_{,\theta\theta}) \\
& - \frac{1}{r} \left( 2U\beta_{,\theta} + U\cot \theta + U_{,\theta} + V\beta_{,rr} - V_{,r}\beta_{,r} - \frac{V_{,rr}}{2} \right) \\
& - r^2e^{-2(\beta-\gamma)} \left[ UU_{,r} \left( \frac{2}{r} - \beta_{,r} + \gamma_{,r} \right) + \frac{UU_{,rr}}{2} + \frac{U_{,r}^2}{2} \right],
\end{aligned} \tag{73}$$

$$R_{(02)} = {}^{GR}R_{02} + \frac{4l^4}{(2l^4 + r^4)^2}r^2V_{,\theta} + \frac{l^4}{(2l^4 + r^4)r} \left[ V_{,\theta}(\gamma_{,r} - 2\beta_{,r}) - \frac{V_{,r\theta}}{2} \right]$$

$$\begin{aligned}
& - \frac{2l^4}{l^4 + r^4} r^3 U^3 \left[ r\gamma_{,r}^2 - 2\beta_{,r} \right] e^{-4(\beta-\gamma)} + \frac{2(2l^4 + r^4)l^4}{(l^4 + r^4)^2} r^2 U^3 e^{-4(\beta-\gamma)} \\
& - \frac{l^4}{2l^4 + r^4} r^4 U \left[ UU_{,r}(\gamma_{,r} - \beta_{,r}) + U_{,r}^2 + \frac{UU_{,rr}}{2} \right] e^{-4(\beta-\gamma)} \\
& - \left[ \frac{4l^8}{(2l^4 + r^4)^2} + \frac{l^8}{(2l^4 + r^4)(l^4 + r^4)} \right] r^3 U^2 U_{,r} e^{-4(\beta-\gamma)} \\
& - \frac{2l^4}{l^4 + r^4} e^{-2(\beta-\gamma)} r U \left[ r\gamma_{,u}\gamma_{,r} - V\gamma_{,r}^2 + U \cot \theta \right] \\
& - \frac{l^4}{2l^4 + r^4} r \left\{ \frac{U_{,rr}V - UV_{,rr}}{2} - U_{,ur} - UV(\beta - \gamma)_{,rr} \right. \\
& \quad \left. + r \left[ U^2(\beta - \gamma)_{,\theta} \right]_{,r} + 2r \left[ U(\beta - \gamma)_{,u} \right]_{,r} \right. \\
& \quad \left. - r(UU_{,r\theta} + U_{,r}U_{,\theta}) - (U_{,r}V + UV_{,r})(\beta - \gamma)_{,r} \right\} e^{-2(\beta-\gamma)} \\
& + \frac{l^4}{(2l^4 + r^4)^2} e^{-2(\beta-\gamma)} \left[ (2l^4 + 5r^4)UV\gamma_{,r} - (2l^4 - r^4)U_{,r}V - 4r^5U^2\gamma_{,\theta} \right] \\
& + \frac{l^4}{(l^4 + r^4)(2l^4 + r^4)^2} \left[ (4l^8 - 3r^8)(2rU\gamma_{,u} + rU_{,u} - 2rU\beta_{,u}) \right. \\
& \quad \left. - (10l^8 + 15l^4r^4 + 7r^8)UV\beta_{,r} + (8l^8 + 4l^4r^4 - r^8)UV_{,r} \right. \\
& \quad \left. - r(4l^8 + 6l^4r^4 + 3r^8)UU_{,\theta} \right] e^{-2(\beta-\gamma)} \\
& - \frac{l^8(13r^8 - 2l^8 + 19l^4r^4)}{r(l^4 + r^4)(2l^4 + r^4)^2} UV, \tag{74}
\end{aligned}$$

$$\begin{aligned}
{}^{GR}R_{02} = & \beta_{,u\theta} - \gamma_{,u\theta} + 2\gamma_{,u}(\gamma_{,\theta} - \cot \theta) - U(\beta_{,\theta\theta} + 2\beta_{,\theta}^2 - 2\beta_{,\theta}\gamma_{,\theta} + \beta_{,\theta} \cot \theta) \\
& - e^{-2(\beta-\gamma)}(UV\gamma_{,r} + UV_{,r} + 2U_{,r}V) - \frac{V_{,r\theta}}{2r} - (\beta_{,r} - \gamma_{,r})\frac{V_{,\theta}}{r} + \frac{V_{,\theta}}{2r^2} \\
& + re^{-2(\beta-\gamma)} \left[ U(3U_{,\theta} + 2\gamma_{,r} - V\gamma_{,rr} - V_{,r}\gamma_{,r}) + U^2(2\gamma_{,\theta} + \cot \theta) \right. \\
& \left. - \frac{U_{,rr}V}{2} + U_{,r}V(\beta_{,r} - \gamma_{,r}) \right] + r^2 e^{-2(\beta-\gamma)} \left[ \frac{U_{,ur}}{2} + U_{,r}U_{,\theta} \right. \\
& \left. - U_{,r}(\beta_{,u} - \gamma_{,u}) + U \left( \frac{3}{2}U_{,r\theta} + 2\gamma_{,ur} + U_{,\theta}\gamma_{,r} + \frac{U_{,r}}{2} \cot \theta \right) \right. \\
& \left. - UU_{,r}(\beta_{,\theta} - \gamma_{,\theta}) + U^2(2\gamma_{,r\theta} + \gamma_{,r} \cot \theta) \right] \\
& - \frac{1}{2}r^4 e^{-4(\beta-\gamma)} UU_{,r}^2, \tag{75}
\end{aligned}$$

$$R_{11} = {}^{GR}R_{11} - \frac{4l^4}{l^4 + r^4} \left( \frac{\gamma_{,r}^2}{2} - \frac{\beta_{,r}}{r} \right) - \frac{4l^4r^2}{(l^4 + r^4)^2}, \tag{76}$$

$${}^{GR}R_{11} = 4 \left( \frac{\gamma_{,r}^2}{2} - \frac{\beta_{,r}}{r} \right), \tag{77}$$

$$R_{(12)} = {}^{GR}R_{12} + \frac{l^4}{l^4 + r^4} e^{-2(\beta-\gamma)} [2rU\gamma_{,r}(r\gamma_{,r} + 1) - 3U(2r\beta_{,r} + 1) + rU_{,r}], \tag{78}$$

$$\begin{aligned}
{}^{GR}R_{12} = & 2 \left[ \gamma_{,r}(\cot \theta - \gamma_{,\theta}) + \frac{\beta_{,\theta}}{r} \right] - \beta_{,r\theta} + \gamma_{,r\theta} \\
& + r^2 e^{-2(\beta-\gamma)} \left( U_{,r}\gamma_{,r} - U_{,r}\beta_{,r} + \frac{2U_{,r}}{r} + \frac{U_{,rr}}{2} \right), \tag{79}
\end{aligned}$$

$$\begin{aligned}
R_{22} = {}^{GR}R_{22} & - \frac{6l^8}{(l^4 + r^4)^2} U^2 r^2 e^{-4(\beta-\gamma)} \\
& + \frac{2l^4}{l^4 + r^4} r^2 \left[ U^2 (r^2 \gamma_{,r}^2 - 2r\beta_{,r} - 1) - rUU_{,r} \right] e^{-4(\beta-\gamma)} \\
& + \frac{2l^4}{l^4 + r^4} \left[ r(U\gamma_{,\theta} + U_{,\theta} + \gamma_{,u}) - V\gamma_{,r} - \frac{V}{r} \right] e^{-2(\beta-\gamma)}, \tag{80}
\end{aligned}$$

$$\begin{aligned}
{}^{GR}R_{22} = & 1 + 2 \left( \beta_{,\theta}\gamma_{,\theta} - \beta_{,\theta}^2 - \gamma_{,\theta}^2 - \beta_{,\theta\theta} + \frac{\gamma_{,\theta\theta}}{2} \right) + 3\gamma_{,\theta} \cot \theta - e^{-2(\beta-\gamma)} (V\beta_{,r} + V_{,r}) \\
& + r e^{-2(\beta-\gamma)} (2\gamma_{,u} + 2U\gamma_{,\theta} + U \cot \theta + 3U_{,\theta} - V\gamma_{,rr} - V_{,r}\gamma_{,r}) \\
& + r^2 e^{-2(\beta-\gamma)} (2\gamma_{,ur} + U_{,\theta}\gamma_{,r} + U\gamma_{,r} \cot \theta + U_{,r\theta} - 2U\gamma_{,r\theta}) \\
& + r^2 U_{,r}^2 \gamma_{,\theta} - r^4 e^{-4(\beta-\gamma)} \frac{U_{,r}^2}{2}, \tag{81}
\end{aligned}$$

$$\begin{aligned}
R_{33}/\sin^2 \theta = & {}^{GR}R_{33}/\sin^2 \theta \\
& - \frac{2l^4}{l^4 + r^4} \left[ r(U\gamma_{,\theta} - U \cot \theta + \gamma_{,u}) - V\gamma_{,r} + \frac{V}{r} \right] e^{-2(\beta-\gamma)}, \tag{82}
\end{aligned}$$

$$\begin{aligned}
{}^{GR}R_{33}/\sin^2 \theta = & e^{-2(\beta-\gamma)} (V\gamma_{,r} - V_{,r}) \\
& + e^{-4\gamma} \left[ 1 + 2\beta_{,\theta}(\gamma_{,\theta} - \cot \theta) + 3\gamma_{,\theta} \cot \theta + \gamma_{,\theta\theta} - 2\gamma_{,\theta}^2 \right] \\
& + r e^{-2(\beta-\gamma)} (U_{,\theta} - 2\gamma_{,u} - 2U\gamma_{,\theta} + 3U \cot \theta + V\gamma_{,rr} + V_{,r}\gamma_{,r}) \\
& + r^2 [(U_{,r} - U\gamma_{,r}) \cot \theta - U_{,\theta}\gamma_{,r} - U_{,r}\gamma_{,\theta} - 2\gamma_{,ur} - 2U\gamma_{,r\theta}]. \tag{83}
\end{aligned}$$

## C Expanded non-zero affine connection components

$$\begin{aligned}\Gamma_{00}^0 = & -\frac{1}{r^2}(M + cc_{,u}) - \frac{1}{r^3}(N \cot \theta + N_{,\theta}) + \dots \\ & -\frac{2l^4}{r^5} + \frac{l^4}{2r^6} \left[ 7M - \frac{5}{2} \left( \frac{c}{3} + \frac{c_{,\theta} \cot \theta}{4} + \frac{c_{,\theta\theta}}{12} \right) \right] + \dots, \end{aligned} \quad (84)$$

$$\Gamma_{01}^0 = -\frac{2l^2}{r^3} - \frac{2l^4}{r^5} + \dots, \quad (85)$$

$$\begin{aligned}\Gamma_{02}^0 = & \frac{N}{r^2} + \frac{1}{r^3} \left( \frac{3C_{,\theta}}{2} + Nc + 3C \cot \theta \right) + \dots \\ & + \frac{l^2}{r^3} (c_{,\theta} + 2c \cot \theta) - \frac{l^2}{r^4} (2N + cc_{,\theta}) + \dots, \end{aligned} \quad (86)$$

$$\Gamma_{10}^0 = \frac{2l^2}{r^3} - \frac{2l^4}{r^5} + \dots, \quad (87)$$

$$\begin{aligned}\Gamma_{20}^0 = & \frac{N}{r^2} + \frac{1}{r^3} \left( \frac{3C_{,\theta}}{2} + Nc + 3C \cot \theta \right) + \dots \\ & - \frac{l^2}{r^3} (c_{,\theta} + 2c \cot \theta) + \frac{l^2}{r^4} (2N + cc_{,\theta}) + \dots, \end{aligned} \quad (88)$$

$$\begin{aligned}\Gamma_{22}^0 = & c + r - \frac{c^2}{2r} - \frac{1}{r^2} \left( \frac{c^3}{3} + C \right) + \dots \\ & - \frac{l^4}{2r^3} - \frac{l^4}{2r^4} c + \frac{3l^4}{8r^5} c^2 + \dots, \end{aligned} \quad (89)$$

$$\begin{aligned}\Gamma_{33}^0 / \sin^2 \theta = & r - c + \frac{c^2}{2r} - \frac{1}{r^2} \left( \frac{c^3}{3} + C \right) + \dots \\ & - \frac{l^4}{2r^3} + \frac{l^4}{2r^4} c + \frac{3l^4}{8r^5} c^2 + \dots, \end{aligned} \quad (90)$$

$$\begin{aligned}\Gamma_{00}^1 = & -\frac{M_{,u}}{r} + \frac{1}{r^2} \left[ M + cc_{,u}(1 + 4 \cot^2 \theta) - \frac{1}{2}(N \cot \theta + N_{,\theta})_{,u} \right. \\ & + c_{,\theta} c_{,u\theta} + 2(c_{,u} c_{,\theta} + cc_{,u\theta}) \cot \theta \left. \right] + \dots \\ & + \frac{2l^2}{r^5} - \frac{l^4}{2r^5} \left[ M_{,u} + \frac{1}{2} \left( \frac{c}{3} + \frac{c_{,\theta} \cot \theta}{4} + \frac{c_{,\theta\theta}}{12} \right)_{,u} \right] + \dots, \end{aligned} \quad (91)$$

$$\begin{aligned}\Gamma_{01}^1 = & \frac{M}{r^2} + \frac{1}{r^3}(N \cot \theta + N_{,\theta}) + \dots \\ & + \frac{2l^2}{r^3} - \frac{4l^2}{r^4} M + \frac{2l^4}{r^5} + \dots, \end{aligned} \quad (92)$$

$$\begin{aligned}\Gamma_{02}^1 = & \frac{1}{r} [c_{,u}(c_{,\theta} + 2c \cot \theta) - M_{,\theta}] \\ & - \frac{1}{2r^2} [2c_{,u}(cc_{,\theta} + 2N) + N + N_{,\theta\theta} + (N_{,\theta} - N \cot \theta) \cot \theta] + \dots \\ & - \frac{l^2}{r^2} (c_{,\theta} + 2c \cot \theta)_{,u} \\ & + \frac{l^2}{r^3} [cc_{,u\theta} + 2c \cot \theta (c_{,u} - 2) + 2c_{,\theta} (c_{,u} - 1) + 2N_{,u} + M_{,\theta}] + \dots, \end{aligned} \quad (93)$$

$$\begin{aligned}\Gamma_{10}^1 = & \frac{M}{r^2} + \frac{1}{r^3}(N \cot \theta + N_{,\theta}) + \dots \\ & - \frac{2l^2}{r^3} + \frac{4l^2}{r^4}M + \frac{2l^4}{r^5} + \dots,\end{aligned}\tag{94}$$

$$\Gamma_{11}^1 = \frac{c^2}{r^3} + \dots + \frac{2l^4}{r^5} + \dots,\tag{95}$$

$$\begin{aligned}\Gamma_{12}^1 = & \frac{1}{r}(c_{,\theta} + 2c \cot \theta) - \frac{1}{r^2}(3N + 3cc_{,\theta} - 2c^2 \cot \theta) + \dots \\ & + \frac{l^2}{r^4}(cc_{,\theta} + 2c^2 \cot \theta) - \frac{l^2}{r^5}c(cc_{,\theta} + 2N) + \dots,\end{aligned}\tag{96}$$

$$\begin{aligned}\Gamma_{20}^1 = & \frac{1}{r}[c_{,u}(c_{,\theta} + 2c \cot \theta) - M_{,\theta}] \\ & - \frac{1}{2r^2}[2c_{,u}(cc_{,\theta} + 2N) + N + N_{,\theta\theta} + (N_{,\theta} - N \cot \theta) \cot \theta] + \dots \\ & + \frac{l^2}{r^2}(c_{,\theta} + 2c \cot \theta)_{,u} \\ & - \frac{l^2}{r^3}[cc_{,u\theta} + 2c \cot \theta(c_{,u} - 2) + 2c_{,\theta}(c_{,u} - 1) + 2N_{,u} + M_{,\theta}] + \dots,\end{aligned}\tag{97}$$

$$\begin{aligned}\Gamma_{21}^1 = & \frac{1}{r}(c_{,\theta} + 2c \cot \theta) - \frac{1}{r^2}(3N + 3cc_{,\theta} - 2c^2 \cot \theta) + \dots \\ & - \frac{l^2}{r^4}(cc_{,\theta} + 2c^2 \cot \theta) + \frac{l^2}{r^5}c(cc_{,\theta} + 2N) + \dots,\end{aligned}\tag{98}$$

$$\begin{aligned}\Gamma_{22}^1 = & -r(1 - c_{,u}) + 2M - 2c_{,\theta} \cot \theta + c(1 + 2c_{,u} + 2 \cot^2 \theta) - c_{,\theta\theta} + \dots \\ & - \frac{l^4}{2r^3}c_{,u} + \dots,\end{aligned}\tag{99}$$

$$\begin{aligned}\Gamma_{33}^1/\sin^2 \theta = & -r(1 + c_{,u}) + 2M + c(1 + 2c_{,u} - 2 \cot^2 \theta) - c_{,\theta} \cot \theta + \dots \\ & + \frac{l^4}{2r^3}c_{,u} + \dots,\end{aligned}\tag{100}$$

$$\begin{aligned}\Gamma_{00}^2 = & \frac{1}{r^2}(c_{,\theta} + 2c \cot \theta)_{,u} \\ & - \frac{1}{r^3}(2N_{,u} + 3cc_{,u\theta} + M_{,\theta} + 4cc_{,u} \cot \theta + c_{,u}c_{,\theta}) + \dots \\ & - \frac{4l^4}{3r^6}(c_{,\theta} + 2c \cot \theta)_{,u} + \dots,\end{aligned}\tag{101}$$

$$\Gamma_{01}^2 = \frac{N}{r^4} + \dots + \frac{l^2}{r^5}(c_{,\theta} + 2c \cot \theta)_{,u} + \dots,\tag{102}$$

$$\begin{aligned}\Gamma_{02}^2 = & \frac{c_{,u}}{r} - \frac{1}{2r^3}(c^2c_{,u} - 2C_{,u}) + \dots \\ & + \frac{l^2}{r^3}(c_{,u} - 1) + \frac{l^2}{r^4}(2M + c) + \dots,\end{aligned}\tag{103}$$

$$\Gamma_{10}^2 = \frac{N}{r^4} + \dots - \frac{l^2}{r^5}(c_{,\theta} + 2c \cot \theta)_{,u} + \dots,\tag{104}$$

$$\begin{aligned}\Gamma_{12}^2 = & \frac{1}{r} - \frac{c}{r^2} - \frac{1}{2r^4}(6C - c^2) + \dots \\ & - \frac{l^2}{r^3} + \frac{l^2}{r^4}c + \dots,\end{aligned}\tag{105}$$



$$\begin{aligned}\Gamma_{20}^2 = & \frac{c,u}{r} - \frac{1}{2r^3}(c^2 c_{,u} - 2C_{,u}) + \dots \\ & - \frac{l^2}{r^3}(c_{,u} - 1) - \frac{l^2}{r^4}(2M + c) + \dots,\end{aligned}\tag{106}$$

$$\begin{aligned}\Gamma_{21}^2 = & \frac{1}{r} - \frac{c}{r^2} - \frac{1}{2r^4}(6C - c^2) + \dots \\ & + \frac{l^2}{r^3} - \frac{l^2}{r^4}c + \dots,\end{aligned}\tag{107}$$

$$\begin{aligned}\Gamma_{22}^2 = & - \frac{2}{r}c \cot \theta + \frac{2}{r^2}(N + cc_{,\theta} + c^2 \cot \theta) + \dots \\ & - \frac{l^4}{6r^5}(c_{,\theta} + 2c \cot \theta) + \dots,\end{aligned}\tag{108}$$

$$\begin{aligned}\Gamma_{33}^2 = & - \sin \theta \cos \theta \left(1 - \frac{2c}{r} - \frac{2}{r^2}N \cot \theta + \dots\right) \\ & - \frac{l^4}{3r^5} \sin^2 \theta (c_{,\theta} + 2c \cot \theta) + \dots,\end{aligned}\tag{109}$$

$$\begin{aligned}\Gamma_{03}^3 = & - \frac{c,u}{r} - \frac{C_{,u}}{r^3} + \frac{c^2 c_{,u}}{2r^3} + \dots \\ & - \frac{l^2}{r^3}(c_{,u} + 1) + \frac{l^2}{r^4}(2M - c) + \dots,\end{aligned}\tag{110}$$

$$\begin{aligned}\Gamma_{13}^3 = & \frac{1}{r} + \frac{c}{r^2} + \frac{3C}{r^4} - \frac{c^3}{2r^4} + \dots \\ & - \frac{l^2}{r^3} - \frac{l^2}{r^4}c + \dots,\end{aligned}\tag{111}$$

$$\begin{aligned}\Gamma_{23}^3 = & \cot \theta - \frac{c_{,\theta}}{r} - \frac{C_{,\theta}}{r^3} + \frac{c^2 c_{,\theta}}{2r^3} + \dots \\ & + \frac{l^2}{r^3}(c_{,\theta} + 2c \cot \theta) + \frac{2l^2}{r^4}(N + c^2 \cot \theta) + \dots,\end{aligned}\tag{112}$$

$$\begin{aligned}\Gamma_{30}^3 = & - \frac{c,u}{r} - \frac{C_{,u}}{r^3} + \frac{c^2 c_{,u}}{2r^3} + \dots \\ & + \frac{l^2}{r^3}(c_{,u} + 1) - \frac{l^2}{r^4}(2M - c) + \dots,\end{aligned}\tag{113}$$

$$\begin{aligned}\Gamma_{31}^3 = & \frac{1}{r} + \frac{c}{r^2} + \frac{3C}{r^4} - \frac{c^3}{2r^4} + \dots \\ & + \frac{l^2}{r^3} + \frac{l^2}{r^4}c + \dots,\end{aligned}\tag{114}$$

$$\begin{aligned}\Gamma_{32}^3 = & \cot \theta - \frac{c_{,\theta}}{r} - \frac{C_{,\theta}}{r^3} + \frac{c^2 c_{,\theta}}{2r^3} + \dots \\ & - \frac{l^2}{r^3}(c_{,\theta} + 2c \cot \theta) - \frac{2l^2}{r^4}(N + c^2 \cot \theta) + \dots,\end{aligned}\tag{115}$$

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